ON PERTURBATIONS OF IDEAL COMPLEMENTS

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To Nigel Kalton on occasion of his 60th birthday.

Abstract. Let $\mathbb{F}[x]$ be the space of polynomials in d variables, let $\mathfrak{G}^N$ be the Grassmannian of N-dimensional subspaces of $\mathbb{F}[x]$ and let J_N stand for the family of all ideals in $\mathbb{F}[x]$ of codimension N. For a given $G \in \mathfrak{G}^N$ we let

$$\mathfrak{J}_G := \{J \in \mathfrak{J}_N : J \cap G = \{0\}\}$$

Is it true, that (with appropriate topology on $\mathfrak{J}_N$) the set $\mathfrak{J}_G$ is dense in $\mathfrak{J}_N$? In general the answer is "No". What is even more surprising, that there are "good ideals" $J \in \mathfrak{J}_N$ such that every "neighborhood" $\mathcal{U}(J) \subset \mathfrak{J}_N$ has a non-empty intersection with $\mathfrak{J}_G$ for any $G \in \mathfrak{G}^N$ and there are "bad" ideals $J \in \mathfrak{J}_N$ (for $d \geq 3$) such that some "neighborhoods" $\mathcal{U}(J) \subset \mathfrak{J}_N$ have an empty intersection with $\mathfrak{J}_G$ for some $G \in \mathfrak{G}^N$. This contrast illuminates the non-homogeneous nature of $\mathfrak{J}_N$.

1. Preliminaries

Let F be a normed linear space, let $\mathfrak{G}^N(F)$ be the Grassmannian of N-dimensional subspaces of F and let $\mathfrak{G}_N(F)$ denotes the Grassmannian of all subspaces of F of codimension N. For a given $G \in \mathfrak{G}_N(F)$ let

$$\mathfrak{G}_G(F) := \{J \in \mathfrak{G}_N(F) : J \cap G = \{0\}\}$$

i.e., $\mathfrak{G}_G(F)$ is a family of all subspaces J in F that are "missing" G or, equivalently, the family of all subspaces of F that complement G.

It is well-known and easy to see that (with appropriate topology on $\mathfrak{G}_N(F)$) for any $G \in \mathfrak{G}^N(F)$, the set $\mathfrak{G}_G(F)$ is an open and dense subset of $\mathfrak{G}_N(F)$.

The main focus of this article is an investigation of the following "ideal" version of this statement.

Let $F = \mathbf{F}[x] = [x_1, ..., x_d]$ be the space of polynomials in d variables over the field $\mathbb{F}$ of real or complex numbers. Let J_N stand for the family of all ideals in $\mathbb{F}[x]$ of codimension N. Let $\mathfrak{G}^N := \mathfrak{G}^N(\mathbb{F}[x])$. For a given $G \in \mathfrak{G}^N$ we let

$$\mathfrak{J}_G := \{J \in \mathfrak{J}_N : J \cap G = \{0\}\}$$

Question : *Is it still true, that* $\mathfrak{J}_G$ is dense in $\mathfrak{J}_N$?

In general the answer is "No". What is even more surprising, that there are "good ideals" $J \in \mathfrak{J}_N$ such that every "neighborhood" $\mathcal{U}(J) \subset \mathfrak{J}_N$ has a non-empty intersection with $\mathfrak{J}_G$ for any $G \in \mathfrak{G}^N$ and there are "bad" ideals $J \in \mathfrak{J}_N$ (for $d \geq 3$) such that some "neighborhoods" $\mathcal{U}(J) \subset \mathfrak{J}_N$ have an empty intersection with $\mathfrak{J}_G$ for some $G \in \mathfrak{G}^N$. This contrast illuminates the non-homogeneous nature of $\mathfrak{J}$ (as oppose to $\mathfrak{G}_N$).

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We will use the rest of this section to describe the topology on $\mathfrak{J}_N$. Actually, any norm on $\mathbb{F}[\mathbf{x}]$ and any "reasonable" notion of convergence of a sequences of ideals in $\mathfrak{J}_N$ will lead to the same results. The underlined reason for it is the existence of intrinsic (Zariski) topology on the Hilbert scheme $\mathfrak{F}^N(\mathbb{F}^d)$ parametrizing $\mathfrak{J}_N$, which is formally weaker then any reasonable topology (cf. [6]).

For the sake of specificity, define the norm of $f = \sum \hat{f}(\mathbf{k})\mathbf{x}^{\mathbf{k}} \in \mathbb{F}[\mathbf{x}]$ by

$$\|f\| := \sum_{\mathbf{k}} \left|\hat{f}(\mathbf{k})\right|,$$

which turns $\mathbb{F}[\mathbf{x}]$ into a normed space with continuous multiplication. Let $(\mathbb{F}[\mathbf{x}])'$ be the dual of $\mathbb{F}[\mathbf{x}]$. An ideal $J \in \mathfrak{J}_N$ induces an N-dimensional subspace $J^{\perp} \subset (\mathbb{F}[\mathbf{x}])'$ defined as

$$J^{\perp} := \{\lambda \in (\mathbb{F}[\mathbf{x}])' : \lambda(f) = 0, \forall f \in J\}$$

that uniquely identifies the ideal J via $(J^{\perp})^{\top} = J$. We will adopt the following definition of convergence:

Definition 1.1. *Let $(J_m,\, m \in \mathbb{N})$ be a sequence of ideals in $\mathfrak{J}_N$ and let $J \in \mathfrak{J}_N$. We say that $J_m \to J$ if for every $\lambda \in J^{\perp}$ there exists $\lambda_m \in J_m^{\perp}$ such that*

$$\lambda_m(f) \longrightarrow \lambda(f) \tag{1.1}$$

for every $f \in \mathbb{F}[\mathbf{x}]$.

A simple perturbation argument yields the following:

Theorem 1.2. *(cf [8]) Let $(J, J_m,\, m \in \mathbb{N})$ be a sequence of ideals in $\mathfrak{J}_N$ such that $J_m \to J$. Let $E \in \mathfrak{G}^N$ complements J. Then the space E complements J_m for sufficiently large m and*

$$P_m f \to P f \tag{1.2}$$

for all $f \in \mathbb{F}[\mathbf{x}]$, where P_m and P are projections onto E with $\ker P_m = J_m$, $\ker P = J$.

Conversely, let P_m and P be projections onto a space E with ideal kernels, such that (1.2) holds. Then $\ker P_m \to \ker P$.

Definition 1.3. *(Birkhoff, [1]). A linear idempotent operator P on $\mathbb{F}[\mathbf{x}]$ is called an ideal projection if $\ker P$ is an ideal in $\mathbb{F}[\mathbf{x}]$.*

The symbol $\mathfrak{P}_N$ will stand for all N-dimensional ideal projections and for a $G \in \mathfrak{G}^N$ we let $\mathfrak{P}_G$ be the family of all ideal projections onto G. Thus $\mathfrak{J}_G$ is in one-to-one correspondence with $\mathfrak{P}_G$.

A nice characterization of ideal projections is due to C. de Boor [2]:

Proposition 1.4. *Let P be a linear mapping on $\mathbb{F}[\mathbf{x}]$. Then P is an ideal projection if and only if*

$$P(fg) = P(f \cdot Pg) \tag{1.3}$$

for all $f, g \in \mathbb{F}[\mathbf{x}]$.

In one variable, the space $F_{<N}[x]$ of polynomials of degree less than N complements every ideal $J \in \mathfrak{J}_N$ (cf. proof of Proposition 2.1 below). In two or more variables there does not exist a subspace $G \in \mathfrak{G}^N$ with this property. However:

Theorem 1.5. *For every N and d there exists a fixed finite family $\mathcal{E}^N$ of N-dimensional (translation-invariant and spanned by monomials) subspaces of $\mathbb{F}[\mathbf{x}]$ such that every $J \in \mathfrak{J}_N$ complements at least one $E \in \mathcal{E}^N$.*

Let $\mathbb{F}_{<N}[\mathbf{x}]$ be the space of polynomials of degree less than N and $\mathbb{F}_{\leq N}[\mathbf{x}]$ be the space of polynomials of degree at most N. It follows from Theorem 1.5 that every $E \in \mathcal{E}^N$ is a subspace of $\mathbb{F}_{<N}[\mathbf{x}]$, i.e., $\mathcal{E}^N \subset \mathfrak{G}^N(\mathbb{F}_{<N}[\mathbf{x}])$

The next theorem is a consequence of Theorem 1.5 and the de Boor's formula.

Theorem 1.6. *Let $(J, J_m,\, m \in \mathbb{N})$ be a sequence of ideals in $\mathfrak{J}_N$ .Then $J_m \to J$ if and only if there exists a subspace $E \in \mathbb{F}_{<N}[\mathbf{x}]$ and a sequence of ideal projections $(P, P_m,\, m > M)$ such* $\ker P = J$, $\ker P_m = J_m$ *and*

$$\|P - P_m\|_{\mathbb{F}_{\leq N}[\mathbf{x}]} \to 0. \tag{1.4}$$

Proof. If $J \in \mathfrak{J}_N$ then by Theorem 1.5 there exists a subspace $E \subset \mathbb{F}_{<N}[\mathbf{x}]$ that complements J and since $\mathbb{F}_{\leq N}[\mathbf{x}]$ is finite-dimensional, by the Theorem 1.2, (1.3) implies (1.4).

Conversely, suppose that $f \in \mathbb{F}_{\leq K}[\mathbf{x}]$ for some $K \in \mathbb{N}$. If $K \leq N$ then (1.2) follows from (1.4). Assuming (1.2) for all monomials $f \in \mathbb{F}_{\leq K}[\mathbf{x}]$, let g be a monomial of degree $K+1$. Then $g = x_j f$ for some $j = 1, ..., d$ and by (1.3):

$$\|Pg - P_m g\|_{\mathbb{F}_{\leq N}[\mathbf{x}]} = \|P(x_j Pf) - P_m(x_j P_m f)\|_{\mathbb{F}_{\leq N}[\mathbf{x}]} \to 0$$

since $x_j Pf, x_j P_m f \in \mathbb{F}_{\leq N}[\mathbf{x}]$ and $x_j P_m f \to x_j Pf$ by inductive assumption. Hence (1.2) holds for all monomials. □

This theorem allows us to define an ε-neighborhood of an ideal $J \in \mathfrak{J}_N$: Let $E \in \mathcal{E}^N$ be such that $J \in \mathfrak{J}_E$. Let P be an ideal projection onto E with $\ker P = E$. Define

$$\mathcal{U}(E, J, \varepsilon) = \{\ker Q, : Q \in \mathfrak{P}_E, \|P - Q\|_{\mathbb{F}_{\leq N}[\mathbf{x}]} < \varepsilon$$

and

$$\mathcal{U}(J, \varepsilon) = \bigcap_{J \in \mathfrak{J}_E} U(E, J, \varepsilon).$$

2. "Good Ideals"

We will now examine closely one special space $E \in \mathcal{E}^N$:

$$E := \operatorname{span}\{1, x_1, x_1^2, ..., x_1^{N-1}\} \tag{2.1}$$

viewed as an N-dimensional subspace of $\mathbb{F}[\mathbf{x}] = \mathbb{F}[x_1, x_2, ..., x_d]$. Let P be an ideal projection onto E. Then

$$\begin{aligned} P(x_1^N) = p_1 &= \textstyle\sum_{j=0}^{N-1} b_{1,j} x_1^j, \\ P(x_2) = p_2 &= \textstyle\sum_{j=0}^{N-1} b_{2,j} x_1^j, \\ &\vdots \\ P(x_d) = p_d &= \textstyle\sum_{j=0}^{N-1} b_{d,j} x_1^j. \end{aligned} \tag{2.2}$$

Let B be the collection of coefficients

$$B = (b_{k,j}, k = 1, ..., d; j = 0, ..., N-1) \tag{2.3}$$

Then, by the de Boor's formula (1.3), we have

$$\begin{aligned} P(x_1^{N+1}) &= P(x_1 P x_1^N) = P(x_1(\sum_{j=0}^{N-1} b_{1,j} x_1^j)) \\ &= P(\sum_{j=0}^{N-1} b_{1,j} x_1^{j+1}) = b_{1,N-1}(\sum_{j=0}^{N-1} b_{1,j} x_1^j + \sum_{j=0}^{N-2} b_{1,j} x_1^{j+1}) \\ &= b_{1,N-1} b_{1,0} + \sum_{j=1}^{N-1} b_{1,N-1}(b_{1,j} + b_{1,j-1}) x_1^j. \end{aligned}$$

Inductively, we conclude that

$$P(x_1^k) = \sum_{j=0}^{N-1} q_{k,j}(B) x_1^j$$

for all k, where $q_{k,j} \in \mathbb{F}[B]$ are polynomials in dN variables B. Now, using $P(x_j f) = P(fPx_j)$ for every $f \in E$ we conclude that

$$Pf = \sum_{j=0}^{N-1} q_{f,j}(B) x_1^j \tag{2.4}$$

for $f \in x_j E$, where $q_{f,j} \in \mathbb{F}[B]$. Inductively, (1.8) holds for all $f \in \mathbf{x}^{\mathbf{k}} E$ and therefore for all $f \in \mathbb{F}[\mathbf{x}]$ with $q_{f,j} \in \mathbb{F}[B]$.

Hence a sequence of d polynomials $(p_1, ..., p_d)$ given by (1.4) (or equivalently the sequence of dN coefficients B) completely determines the ideal projector P onto E. What so special about this particular space E is that the converse also holds:

Proposition 2.1. *Every sequence $(p_1, ..., p_d)$ of polynomials in E defines an ideal $J = \langle x_1^N - p_1, x_2 - p_2, ..., x_d - p_d \rangle$ that complements E. Hence every sequence B of dN scalars defines an ideal projection P_B onto E by (2.4) and every ideal projection P onto E defines a sequence B_P by (2.2). Clearly*

$$P_{B_P} = P \text{ and } B_{P_B} = B.$$

Proof. It is clear from the construction of E that $E \cap J = \{0\}$. Let $f \in \mathbb{F}[x_1]$ be a polynomial in only one variable. Using the division algorithm in $\mathbb{F}[x_1]$ we have $f = q(x_1^N - p_1) + r$ with $\deg r < N$. Thus the ideal $\langle x_1^N - p_1 \rangle$ generated by $x_1^N - p_1$ complements the space $\mathbb{F}_{<N}[x_1]$ of polynomials of degree less than N in $\mathbb{F}[x_1]$ and $E + J \supset \mathbb{F}[x_1]$. Inductively, we assume that $E + J \supset \mathbb{F}[x_1, ..., x_k]$, $k < d$ and prove that $E + J \supset \mathbb{F}[x_1, ..., x_k, x_{k+1}]$, i.e., we need to show that $x_{k+1}^n \mathbb{F}[x_1, ..., x_k] \subset E + J$ for all n. For $f \in \mathbb{F}[x_1, ..., x_k]$ we have

$$x_{k+1} f = (x_{k+1} - p_{k+1}) f + p_{k+1} f \in E + J \tag{2.5}$$

since the first term is in the ideal J and the second belongs to $E + J$ by inductive assumption. Using induction on n, assume that $f \in x_{k+1}^n \mathbb{F}[x_1, ..., x_k]$ and conclude that $x_{k+1} f \in x_{k+1}^{n+1} \mathbb{F}[x_1, ..., x_k]$ has a representation (2.5). □

Corollary 2.2. *$B_m \to B$ if and only if $\ker P_{B_m} \to \ker P_B$.*

Proof. Since $q_{f,j}(B)$ in (2.4) are polynomials (hence continuous function of B) $B_m \to B$ implies $P_{B_m} f \to P_B f$ for every f. Conversely, if $P_{B_m} f \to P_B f$ then

$P_{B_m}x_1^N \to P_B x_1^N$ and $P_{B_m}x_j \to P_B x_j$ for all $j = 2, ..., d$. Now $B_m \to B$ follows from (2.2). $\square$

We are now ready to prove that every ideal $J \in \mathfrak{J}_E$ is a "good ideal".

Theorem 2.3. *Let $J \in \mathfrak{J}_E$ and $G \in \mathfrak{G}^N$. Then every neighborhood $\mathcal{U}(J)$ has a non-empty intersection with $\mathfrak{J}_G$.*

Proof. First, we establish that $\mathfrak{J}_E \cap \mathfrak{J}_G \neq \varnothing$. Let $g_1, ..., g_N$ be a bases in G and $e_1, ..., e_N$ be a bases in E. There exists a sequence $\{\mathbf{z}_1^*, ..., \mathbf{z}_N^*\}$ of points in $\mathbb{F}^d$ such that $g_j(\mathbf{z}_k^*) = \delta_{j,k}$ and hence the polynomial

$$\det(g_j(\mathbf{z}_k), j = 1, ...N)$$

in dN variables (coordinates of the points $\mathbf{z}_k$) is not identically zero. Therefore the set

$$\mathcal{Z}_G := \{(\mathbf{z}_1, ..., \mathbf{z}_N) : \det(g_j(\mathbf{z}_k)) \neq 0$$

is an open and dense set in $(\mathbb{F}^d)^N$. Similarly the set

$$\mathcal{Z}_E := \{(\mathbf{z}_1, ..., \mathbf{z}_N) : \det(e_j(\mathbf{z}_k)) \neq 0$$

is an open and dense set in $(\mathbb{F}^d)^N$. Thus $\mathcal{Z}_G \cap \mathcal{Z}_E \neq \varnothing$ and for any $(\mathbf{z}_1, ..., \mathbf{z}_N) \in \mathcal{Z}_G \cap \mathcal{Z}_E$, the ideal

$$J := \{f \in \mathbb{F}[\mathbf{x}] : f(\mathbf{z}_j) = 0, j = 1, ...N\}$$

complements E and G.

Therefore, there exists a sequence $B^* \in \mathbb{F}^{dN}$ such that P_{B^*} is an ideal projection onto E and $\ker P_{B^*} \in \mathfrak{J}_G$. Let

$$\mathcal{B}_G := \{B \in \mathbb{F}^{dN} : \ker P_B \in \mathfrak{J}_G\}.$$

It follows that $\mathcal{B}_G \neq \varnothing$. Suppose that $\ker P_B \in \mathfrak{J}_G$ i.e., $\ker P_B \cap G = \{0\}$. Then

$$P_B(\sum_{k=1}^{N} \alpha_k g_k) = \sum_{k=1}^{N} \alpha_k P_B g_k = 0$$

implies $\alpha_k = 0$ for all $k = 1, ..., N$. Hence $\ker P_B \in \mathfrak{J}_G$ is the same as linear independency of the sequence of polynomials $(P_B g_k, k = 1, ..., N)$. Since, by (2.4), $P_B f = \sum_{j=0}^{N-1} q_{g_k,j}(B) x_1^j$ this is equivalent to

$$\det(q_{g_k,j}(B)) \neq 0.$$

Since $\mathcal{B}_G \neq \varnothing$, this determinant is a non-zero polynomial in $\mathbb{F}[B]$, hence there exists an open and dense set of $B \subset \mathbb{F}^{dN}$ such that $\ker P_B \in \mathfrak{J}_G$. $\square$

3. "Bad Ideals"

In this section we will use a beautiful construction of A. Iarrobino and modified the reasoning of [7] to show that for $d \geq 3$ and for sufficiently large N there exists an ideal $J \in \mathfrak{J}_N$ such that J is not the limit of ideals in $\mathfrak{J}_E$, where $E :=\text{span}\{1, x_1, x_1^2, ..., x_1^{N-1}\}$ is the space considered in the previous section.

Let $W := M_{<n}^d[\mathbf{x}]$ be the set of monomials of degree less than n. Let $U \cup V = M_n^d[\mathbf{x}]$ be a non-trivial partition of the set $M_n^d[\mathbf{x}]$ of all monomials of degree n in $\mathbb{F}[\mathbf{x}]$. Let H be the subspace of $\mathbb{F}[\mathbf{x}]$ spanned by $\mathbb{F}_{<n}^d[\mathbf{x}]$ and V and let

$$N = \dim H = \#V + \#W \tag{3.1}$$

For any choice of matrix $C \in \mathbb{F}^{U\times V}$, the space J_C spanned by monomials of degree greater than n and the specific polynomials

$$p_u := u - \sum_{v\in V} C(u,v)v, \ \ u \in U, \tag{3.2}$$

complements H and is an ideal. The latter is so because each p_u is homogeneous, thus every product of a monomial with p_u is in J_C. Furthermore, as is easy to see

$$\mathbb{F}[\mathbf{x}] = H \oplus J_C. \tag{3.3}$$

for each C.

Theorem 3.1. *For any $d \geq 3$ and for sufficiently large n, there exists a partition $U \cup V = M_n^d[\mathbf{x}]$ and a matrix $C \in \mathbb{F}^{U\times V}$ such that the ideal J_C can not be perturbed to complement E.*

Proof. Assume that J_C complements H and that there is a small perturbation of J_C that complements E. In other words, assume the existence of a sequence of ideals $\{J_m\}$ such that each J_m complements G and $J_m \to J_C$. This is the same as the existence of a sequence of ideal projections P_m onto H such that $\ker P_n$ complements E and H at the same time and $P_m f \to Pf$ for every $f \in \mathbb{F}^d[\mathbf{x}]$. In particular

$$u - P_m u \to u - \sum_{v\in V} C(u,v)v, \forall\ u \in U.$$

Since P_m is a projection onto H it follows that

$$P_m u = \sum_{v\in V} C_m(u,v)v + \sum_{w\in W} C_m(u,w)w \tag{3.4}$$

and

$$C_m(u,v) \to C(u,v), \forall u \in U \text{ and } C_m(u,w) \to 0, \forall w \in W. \tag{3.5}$$

Since $\ker P_m$ complements E, we let Q_m be the ideal projection onto E with $\ker Q_m = \ker P_m$. Then $u - P_m u \in \ker P_m = \ker Q_m$ and

$$0 = Q_m(u - P_m u) = Q_m u - (\sum_{v\in V} C_m(u,v)Q_m v + \sum_{w\in W} C_m(u,w)Q_m w)$$

or

$$\sum_{v\in V} C_m(u,v)Q_m v + \sum_{w\in W} C_m(u,w)Q_m w = Q_m u. \tag{3.6}$$

At this point it is important to notice that, as projections onto E, the operators Q_m depend polynomially on $d \times N$ parameters B as in (2,4):

$$Q_m f = \sum_{k=0}^{N-1} q_{k,f}^{(m)}(B) x_1^k \tag{3.7}$$

where $q_{k,f}^{(n)} \in \mathbb{F}^{dN}[B]$. Rewriting (3.6) we have:

$$\sum_{k=0}^{N-1} (\sum_{v\in V} C_m(u,v) q_{k,v}^{(m)}(B) + \sum_{w\in W} C_m(u,W) q_{k,w}^{(m)}(B)) x_1^k = \sum_{k=0}^{N-1} q_{k,u}^{(m)}(B) x_1^k \tag{3.8}$$

or equivalently

(3.9)
$$\sum_{v\in V} C_m(u,v)q^{(m)}_{k,v}(B) + \sum_{w\in W} C_m(u,w)q^{(m)}_{k,w}(B) = q^{(m)}_{k,u}(B), u\in U, k=0,...,N-1.$$

Since $\#V + \#W = N$, this is the system of $N\times\#U$ equations with the same number of unknowns $\{C_m(u,v), u\in U\}$ and $\{C_m(u,w), u\in U\}$. By Cramer's rule

$$C_m(u,v) = \frac{\det V_{m,u,v}(B)}{\det V_m(B)}, \tag{3.10}$$

where $\det V_m(B)$ is the determinant of the matrix on the left-hand side of (3.9), and $\det V_{m,u,v}(A)$ is the determinant of the same matrix with the (u,k)-th column replaced by the column $(q^{(m)}_{k,u}(B))$. Notice that that makes $C_m(u,v)$ rational function of $d\times N$ parameters B with common denominator. Thinking of $1/\det V_m(B)$ as just another variable, say z, we conclude that the set

$$\mathfrak{C} := \{z \det V_{n,u,v}(B),\ B\in\mathbb{F}^{dN}, z\in\mathbb{F}\backslash\{0\}\}\subset\mathbb{F}^{\#U\times\#V}$$

is a polynomial image of $\mathbb{F}^{dN+1}$ where the first dN parameters are the parameters of B. As such $\mathfrak{C}$ forms an affine subvariety of $\mathbb{F}^{\#U\times\#V}$ with $\dim\mathfrak{C} \le dN+1$ (cf [4], Theorem 2, p. 466). Hence if

$$\#U\times\#V > dN+1 \tag{3.11}$$

then there exists a collection $C(u,v)\in\mathbb{F}^{\#U\times\#V}$ which is not in the closure of $\mathfrak{C}$, contradicting (3,5).

Now we only need to count. As was pointed out in [5], we have $\#M^d_n[\mathbf{x}] = \binom{n+d-1}{d-1} \approx (\frac{n^{d-1}}{(d-1)!})$, choosing the partition $U\cup V = M^d_n[\mathbf{x}]$ such that $\#U = \left\lfloor \frac{1}{2}\binom{n+d-1}{d-1}\right\rfloor$ we have

$$\#U\times\#V = \left\lfloor\frac{1}{2}\binom{n+d-1}{d-1}\right\rfloor\left\lceil\frac{1}{2}\binom{n+d-1}{d-1}\right\rceil \approx \frac{1}{4}(\frac{n^{d-1}}{(d-1)!})^2 \approx n^{2d-2}$$

while $N = \dim H = \dim\mathbb{F}^d_{\le n}[\mathbf{x}] - \#U = \binom{n+d}{d} - \left\lfloor\frac{1}{2}\binom{n+d-1}{d-1}\right\rfloor \approx \frac{n^d}{d!} \approx n^d$. Hence for sufficiently large n, (3.11) holds. Direct computation yield (3.11) with $n=7$, for $d=3$; $n=3$ for $d=4$ or 5 and $n=2$ for $d>5$. ☐

Remark 3.2. *The "bad" ideals do not exist in* $\mathbb{F}[\mathbf{x}]$ *for* $d=1$*, as follow from the Theorem2.3, since, for* d = 1*, every ideal in* $\mathfrak{J}_N$ *is complemented to*

$$E := span\{1, x_1, x_1^2, ..., x_1^{N-1}\}.$$

"Bad" ideals also do not exist in $\mathbb{C}[\mathbf{x}]$ *for* $d=2$ *as noted in [8]. The existence of "bad" ideals in* $\mathbb{R}[\mathbf{x}]$ *for* $d=2$ *is an open problem.*

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